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Thermoelastic damping effect on in-extensional vibration of rotating thin ring

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A B S T R A C T

Sensitive devices such as resonant sensors and radio frequency micro-electro-mechanical system (RF-MEMS) filters etc., require high Quality factors (Q-factors) defined as the ratio of total system energy to dissipation that occurs due to various damping mechanisms. Also, thermoelastic damping is considered to be one of the most important factors to elicit energy dissipation due to the irreversible heat flow of oscillating structures in the micro scales. In this study, the Q-factor for thermoelastic damping is investigated in rotating thin rings with in-plane vibration. First, in order to obtain the temperature profile of the model, a heat conduction equation for the thermal flow across the radial direction is solved based on the bending approximation so-called in-extensional approximation of the ring. Using the temperature distribution coupled with a displacement, a governing equation of the ring model can then be derived. Eventually, an eigen-value analysis is performed to obtain the natural frequency of rotating thin rings, and the analytical and numerical values of Q-factors can then be determined by the definition. Furthermore, the effects of rotating speed, dimensions of the ring, mode numbers and ambient temperatures on the Q-factor are discussed in detail.

1. Introduction

Resonators have been developed for rate sensors, namely gyroscopes of micro-electro-mechanical systems (MEMS). Also, the systems have been widely used in aerospace, automotive and industrial parts such as automotive vehicle stability enhancement systems, aerospace control systems and RF-MEMS filters, etc. Energy dissipation due to various damping factors is extremely significant during the operation of rate sensors in working mechanisms. Thus, it is very important to thoroughly examine and fully analyze the influence of damping mechanisms on the oscillating structures.

For a wide range of application fields of resonators, the aim of design is to maximize the Quality factors (Q-factors) defined as the ratio of the kinetic and potential energy of the system to the dissipated energy by various damping mechanisms [1]. Generally, there are two reasons for energy dissipation of the vibrant structures: extrinsic and intrinsic damping mechanisms. Extrinsic damping mechanisms are gas damping, support losses and squeeze-film damping. These mechanisms can be effectively eliminated by suitable handling. For instance, evacuating the environment of oscillating structures, the gas damping effect on the structures can be neglected [2]. Moreover, the support losses originated from the

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energy exchange between the resonator and the support can be minimized by proper design [3]. Finally, squeeze-film damping can be negligible by placing the capacitor electrodes and encapsulating the device from afar [4]. On the other hand, intrinsic mechanisms are closely related to the material and geometric properties of structures. Thus, it is significantly more difficult to minimize the effect of intrinsic mechanisms than extrinsic mechanisms. Up to now, it has been well known that one of the most important energy loss factors is thermoelastic damping in very small structures. Also, previous research [5] has shown that this kind of damping creates energy dissipation by the irreversible heat flow of oscillating structures.

A great deal of work has been carried out on Q-factor for thermoelastic damping of structures. According to the various objectives in each application, many kinds of structures such as beams, thin rings, rectangular and circular plates and cylindrical shells, etc. have been widely adopted in aerospace, automotive and industrial fields. Firstly, Zener [6] presented the analytic and approximate form of the factor for the homogeneous, isotropic and uniform beams based on some additional assumptions. Furthermore, the paper emphasized that the fluctuations of temperature in a vibrating beam are important in Q-factors associated with thermodynamical considerations. Lifshitz and Roukes [7] refined Zener's work for thin beams. Using the equations of linear thermoelasticity, the process of thermoelastic damping, namely fundamental dissipation mechanism in micro- and nanomechanical systems, was examined. Using the flexural vibrating beam model, Duwel et al. [8] compared the theoretical value of the Q-factor to the experimental result, and showed that Zener's method could successfully depict the effects of beam dimensions and material properties on the factor relevant to thermoelastic damping. Khisaeva and Ostoja-Starzewski [9] examined thermoelastic damping in micro-/nano-beams considering the finite speed of heat transfer by a hyperbolic heat conduction equation. Meanwhile, Wong et al. [10] applied Zener's theory to thin silicon rings, and obtained the theoretical expression of Q-factor. Moreover, it was verified that the theoretical and experimental results were almost equal for the practical size of the model. Applying a finite element method, Yi [11] studied the geometric effects on thermoelastic damping in MEMS resonators. To obtain the linear eigenvalue equation, perturbation forms of the temperature and displacements are used. Furthermore, the order of the problem and computational time can be reduced by using the Fourier reduction method. Nayfeh and Younis [5] acquired the analytical form of the Q-factor for thermoelastic damping with rectangular microplates. The perturbation method is also used to obtain the analytical expression of the factor. Applying a thermal-energy approach to the formulation, Hao [12] analytically investigated the damping of micro- and nano-electromechanical circular thin-plate resonators with the contour-mode vibrations. Meanwhile, Lu et al. [13] presented an approximate form of the Q-factor for thermoelastic damping in a thin cylindrical shell. The general thermoelastic coupled equations are simplified by using Donnell–Mushtari–Vlasov approaches and are approximately solved by using the Galerkin method.

In this paper, the Q-factor relevant to thermoelastic damping in the rotating thin ring under the in-plane vibration is investigated. Using the bending approximation in the thin beam theory, the temperature profile of the ring is obtained. The heat conduction equation for the thermal flow in the radial direction is considered. An analytic form of Q-factor is determined using the results of the eigen-value analysis with the temperature distribution. Also, the results are compared with the numerical data created by the iterative method for verification. Furthermore, the influences of the rotating speed, the dimensions of the ring, the eigen-modes and ambient temperatures on the natural frequency and the Q-factor are widely examined.

2. Formulations

In this chapter, a steady rotating thin ring with the angular velocity Ω under thermoelastic damping is considered in order to analyze the Q-factor of the model. Fig. 1 shows a global polar coordinate system (r,θ,Z) and a local coordinate system (x,y,z). Here, the x-, y- and z-axes are radially directed outwards, circumferentially directed, and tangential to the

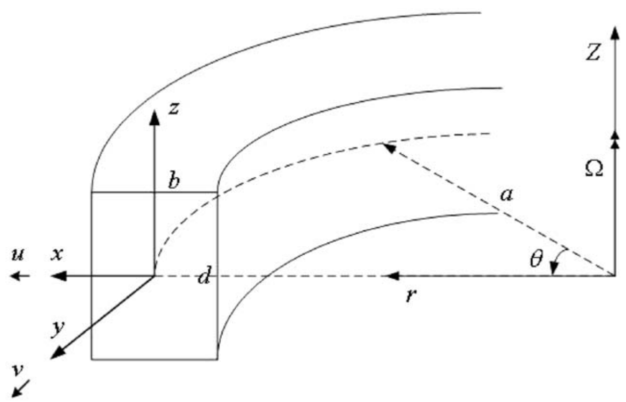

Fig. 1. Geometry of a thin ring with a global and local coordinate system.

cross section, respectively. Also, the z-axis in the local coordinate system is the same as the direction of the z-axis in the global polar coordinate system. In addition, the geometry of a thin ring is depicted in the figure with mean radius a, radial thickness b, and axial depth d. Moreover, u and v denote the displacements in the x and y directions, respectively. Their harmonic solutions can be assumed as

$$u(\theta,\ t) = U_0(\theta)e^{i\omega t} = A_u e^{i(n\theta+\omega t)}$$

$$v(\theta,\ t) = V_0(\theta)e^{i\omega t} = A_v e^{i(n\theta+\omega t)} \tag{1}$$

where mode number $n = 2, 3, 4, \ldots$.

2.1. Heat conduction equation

To obtain the temperature profile for the thermal flow coupled with the strain, the heat conduction equation is employed as [14]

$$\frac{\partial T}{\partial t} - \chi\nabla^2 T = -\frac{E\alpha T_a}{C_v(1-2v)}\left(\frac{\partial\varepsilon}{\partial t}\right) \tag{2}$$

where T is the change in the temperature from the ambient temperature T_a, and can be assumed as

$$T(x,\ \theta,\ t) = T_0(x,\ \theta)e^{i\omega t} \tag{3}$$

Furthermore, χ, C_v and α are the thermal diffusivity of the material, the heat capacity per unit volume, and the coefficient of thermal expansion, respectively. Meanwhile, E, v and ε are the Young's modulus, the Poisson's ratio and the dilatation, respectively.

If the heat flow between the surfaces and the environment of the ring is negligible, then the zero heat flux boundary conditions can be applied. Therefore, the temperature profile of a thin ring coupled with displacement can be obtained as [15]

$$T_0(\theta,\ x) = \frac{\Delta_E}{\alpha a^2}\left(\frac{\partial^2 U_0(\theta)}{\partial\theta^2} + U_0(\theta)\right)\left(x - \frac{\sin kx}{k\cos\frac{bk}{2}}\right) \tag{4}$$

where

$$\Delta_E = \frac{E\alpha^2 T_a}{C_v}, \quad k = (1-i)\sqrt{\frac{\omega}{2\chi}}. \tag{5}$$

Because k depends on natural frequency ω, the temperature distribution T_0 is a function of ω.

2.2. Quality factor of rotating thin rings

Equation of motion for rotating thin rings are given as [16]

$$\frac{1}{a^2}\frac{\partial^3 M_\theta}{\partial\theta^3} + \frac{1}{a^2}\frac{\partial M_\theta}{\partial\theta} - \rho A\left(\frac{\partial^2 v}{\partial t^2} + 2\frac{\partial u}{\partial t}\Omega - v\Omega^2\right) - \rho A\frac{\partial}{\partial\theta}\left[\frac{\partial^2 u}{\partial t^2} - 2\frac{\partial v}{\partial t}\Omega - (a+u)\Omega^2\right] = 0 \tag{6}$$

where ρ, A and Ω are the mass density, the cross sectional area and the angular velocity of the ring, respectively.

In order to express the equation of motion in terms of displacement fields only, the adequate form of bending moment considering the thermal effect on a cross section is derived as [15]

$$M_\theta = -\frac{E_\omega I}{a^2}\left(\frac{\partial^2 u}{\partial\theta^2} + u\right) \tag{7}$$

where the frequency dependent elastic modulus E_ω and the second moment of inertia of the ring cross sectional area I are defined as

$$E_\omega = E[1 + \Delta_E\{1 + f(\omega)\}], \quad I = \frac{db^3}{12} \tag{8}$$

Assuming that the deformation for circumferential centerline of the ring is very small compared to the bending deformation, in-extensional assumption can be applied

$$\frac{\partial^6 u}{\partial\theta^6} + 2\frac{\partial^4 u}{\partial\theta^4} + \frac{\partial^2 u}{\partial\theta^2} + \frac{\rho A a^4}{E_\omega I}\left[\frac{\partial^4 u}{\partial\theta^2\partial t^2} - \frac{\partial^2 u}{\partial t^2} + 4\Omega\frac{\partial^2 u}{\partial\theta\partial t} + \Omega^2\left(u - \frac{\partial^2 u}{\partial\theta^2}\right)\right] = 0. \tag{9}$$

The solution of this equation can be obtained by assuming the displacement u as the form of Eq. (1). As a result, the natural frequency of a rotating thin ring is given as:

$$\omega=\frac{2\Omega n}{1+n^2}\pm\sqrt{\left(\frac{2\Omega n}{1+n^2}\right)^2+\left(\frac{E_\omega I}{\rho A a^4}\right)\frac{(n^2-1)^2n^2}{1+n^2}-\Omega^2} \tag{10}$$

Eq. (10) is implicit of the natural frequency because the frequency dependent elastic modulus E_ω is a function of natural frequency. The Q-factor of the rotating thin ring can be directly obtained using the iterative method from Eq. (10). For various examples, the numerical results will be compared with the analytic solution for Q-factor in the next section.

Furthermore, the isothermal natural frequency of the non-rotating model $\omega_{0,nr}$ can be expressed as follows:

$$\omega_{0,nr}=\frac{(n^2-1)n}{a^2\sqrt{1+n^2}}\sqrt{\frac{EI}{\rho A}}. \tag{11}$$

Now, the following factor is introduced,

$$\xi=b\sqrt{\frac{\omega_{0,nr}}{2\chi}} \tag{12}$$

and complex function $f(\omega)$ can be rewritten as:

$$f(\omega_{0,\,nr})=-\frac{6}{\xi^3}\left(\frac{\sinh\xi-\sin\xi}{\cosh\xi+\cos\xi}\right)+\left\{\frac{6}{\xi^3}\left(\frac{\sinh\xi+\sin\xi}{\cosh\xi+\cos\xi}\right)-\frac{6}{\xi^2}\right\}i. \tag{13}$$

Using Eqs. (10) and (11), the ratio of the natural frequencies between the case of the rotating ring and the isothermal case without a rotation is:

$$\frac{\omega}{\omega_{0,\,nr}}=\left(\frac{2n}{1+n^2}\right)\frac{\Omega}{\omega_{0,\,nr}}\pm\sqrt{1+\Delta_E\{1+f(\omega_{0,\,nr})\}-\frac{(n^2-1)^2}{(1+n^2)^2}\left(\frac{\Omega}{\omega_{0,\,nr}}\right)^2} \tag{14}$$

Assuming that Δ_E and $\Omega/\omega_{0,\,nr}$ are small compared to unity, series expansion can be used to simplify the square root term:

$$\frac{\omega}{\omega_{0,\,nr}}=\left(\frac{2n}{1+n^2}\right)\frac{\Omega}{\omega_{0,\,nr}}\pm\left[1+\frac{\Delta_E}{2}\{1+f(\omega_{0,\,nr})\}-\frac{(n^2-1)^2}{(1+n^2)^2}\left(\frac{\Omega}{\omega_{0,\,nr}}\right)^2\right]. \tag{15}$$

Thus, the real and imaginary parts of a non-dimensional natural frequency are

$$\mathrm{Re}\left(\frac{\omega}{\omega_{0,\,nr}}\right)=\left(\frac{2n}{1+n^2}\right)\frac{\Omega}{\omega_{0,\,nr}}\pm\left[1+\frac{\Delta_E}{2}\left\{1-\frac{6}{\xi^3}\left(\frac{\sinh\xi-\sin\xi}{\cosh\xi+\cos\xi}\right)\right\}-\frac{(n^2-1)^2}{2(1+n^2)^2}\left(\frac{\Omega}{\omega_{0,\,nr}}\right)^2\right] \tag{16}$$

$$\mathrm{Im}\left(\frac{\omega}{\omega_{0,\,nr}}\right)=\pm\frac{\Delta_E}{2}\left\{\frac{6}{\xi^3}\left(\frac{\sinh\xi+\sin\xi}{\cosh\xi+\cos\xi}\right)-\frac{6}{\xi^2}\right\} \tag{17}$$

Finally, the Q-factor for the case of the rotating thin ring yields:

$$Q_r^{-1}=2\left|\frac{\mathrm{Im}(\omega)}{\mathrm{Re}(\omega)}\right|=2\left|\frac{\mathrm{Im}\left(\frac{\omega}{\omega_{0,\,nr}}\right)}{\mathrm{Re}\left(\frac{\omega}{\omega_{0,\,nr}}\right)}\right|=\left|\frac{\Delta_E\left\{\frac{6}{\xi^3}\left(\frac{\sinh\xi+\sin\xi}{\cosh\xi+\cos\xi}\right)-\frac{6}{\xi^2}\right\}}{\left(\frac{2n}{1+n^2}\right)\frac{\Omega}{\omega_{0,\,nr}}\pm\left[1-\frac{(n^2-1)^2}{2(1+n^2)^2}\left(\frac{\Omega^2}{\omega_{0,\,nr}^2}\right)\right]}\right| \tag{18}$$

where the subscript r denotes the rotational case.

3. Results and discussion

In this chapter, a rotating silicon ring is selected to examine the analytic and numerical results for the Q-factor relevant to the thermoelastic damping. As shown in Eq. (18), if the ring has no rotating speed ($\Omega=0$), the expression for Q-factor exactly equals the result in Ref. [15]. The mechanical and thermal properties [10] of silicon in this study are shown in Table 1. Also, since many devices such as the resonator gyro are generally operated at the $n=2$ mode, unless otherwise

Table 1
Mechanical and thermal properties at 298 K [10].

Young's modulus, E	165 GPa
Mass density, ρ	2.33×10^3 Kg m^{-3}
Thermal expansion coefficient, α	2.6×10^{-6} K^{-1}
Heat capacity per unit volume, C_v	1.64×10^6 J m^{-3} K^{-1}
Thermal diffusivity, χ	8.6×10^{-5} m^2 s^{-1}

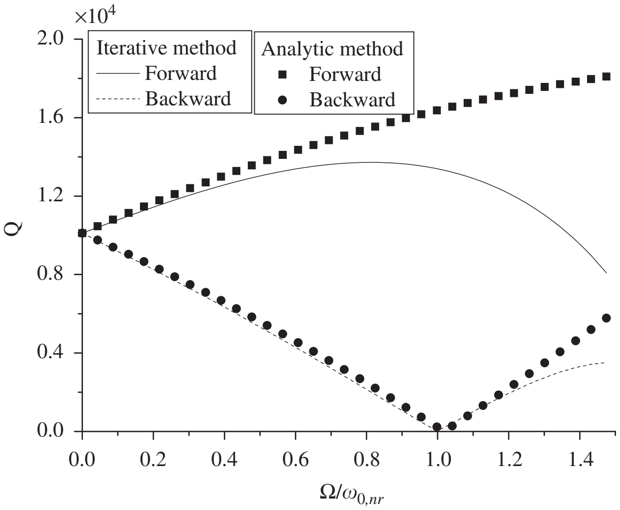

Fig. 2. Q-factor of rotating ring with respect to rotation ($a=3\,\text{mm}, b=100\,\mu\text{m}$).

Table 2
Q-factor for various radius a and thickness b.

a(mm)	b(μm)							
	160		120		80		40	
5	**11,241** −9.765%	10,241	**12,604** −4.450%	12,067	**32,590** −0.061%	32,570	**246,947** 2.922%	254,380
4	**13,307** −8.117%	12,308	**10,724** −5.510%	10,164	**21,761** −0.890%	21,569	**161,009** 1.157%	162,894
3	**19,605** −5.273%	18,623	**11,295** −5.266%	10,730	**13,865** −1.776%	13,623	**91,532** 0.307%	91,814
2	**38,928** −2.456%	37,995	**18,402** −3.104%	17,848	**10,238** −2.513%	9987	**41,331** −0.080%	41,298
1	**139,702** −0.628%	138,830	**61,977** −0.826%	61,469	**20,793** −1.187%	20,549	**12,687** −0.499%	12,624

Bold: rotating case (Ω=4 × 10^4 rpm).
Normal: non-rotating case.

noted, the Q-factor is calculated only for the mode. The ambient temperature T_a is assumed to be 298 K(normal temperature) for typical surroundings.

Fig. 2 represents the effect of rotating speed on the Q-factors. It shows the bifurcations of Q-factors due to the Coriolis acceleration effect. For the iterative method, solid and dashed lines are related to the backward and forward traveling waves, respectively. Meanwhile, square and circle symbols denote the backward and forward waves, respectively, for the analytic method. As shown in the figure, the result of the analytic method for the Q-factor agrees well with that of the iterative method within $\Omega/\omega_{0,nr} \simeq 0.58$ or $\Omega = 4\times 10^5$ rpm. When the rotating speed is low enough, the Q-factor is rarely affected. However, as the rotating speed increases, the rotating effect on the Q-factor definitely increases, especially for the backward traveling wave types. Thus, if the ring has a high rotating speed, the rotating effect should be included.

Table 2 illustrates the comparative Q-factors between a non-rotating case and a rotating case ($\Omega = 4\times 10^4$ rpm) with various dimensions. Bold and normal fonts refer to the rotating and non-rotating cases, respectively. As shown in the data, the Q-factor with the rotating case has somewhat a high value compared to that of the non-rotating case for the most range of given dimensions. However, there is a different tendency only in the small radial thickness $b = 40\,\mu$m. The difference between the two cases is within the range −9.765 percent∼+2.922 percent.

The inverse of the Q-factor for the rotating speed $\Omega = 4\times 10^3$ rpm with different geometries for the thin silicon ring is depicted in Fig. 3(a). In this figure, Q-factors obtained from the analytic method yield almost the same values compared to that of the iterative method. On the design point of view, the ring should not be operated in vibration modes which have a high thermoelastic damping. Meanwhile, when the radial thickness of the ring is decreasing, the magnitudes of the inverse

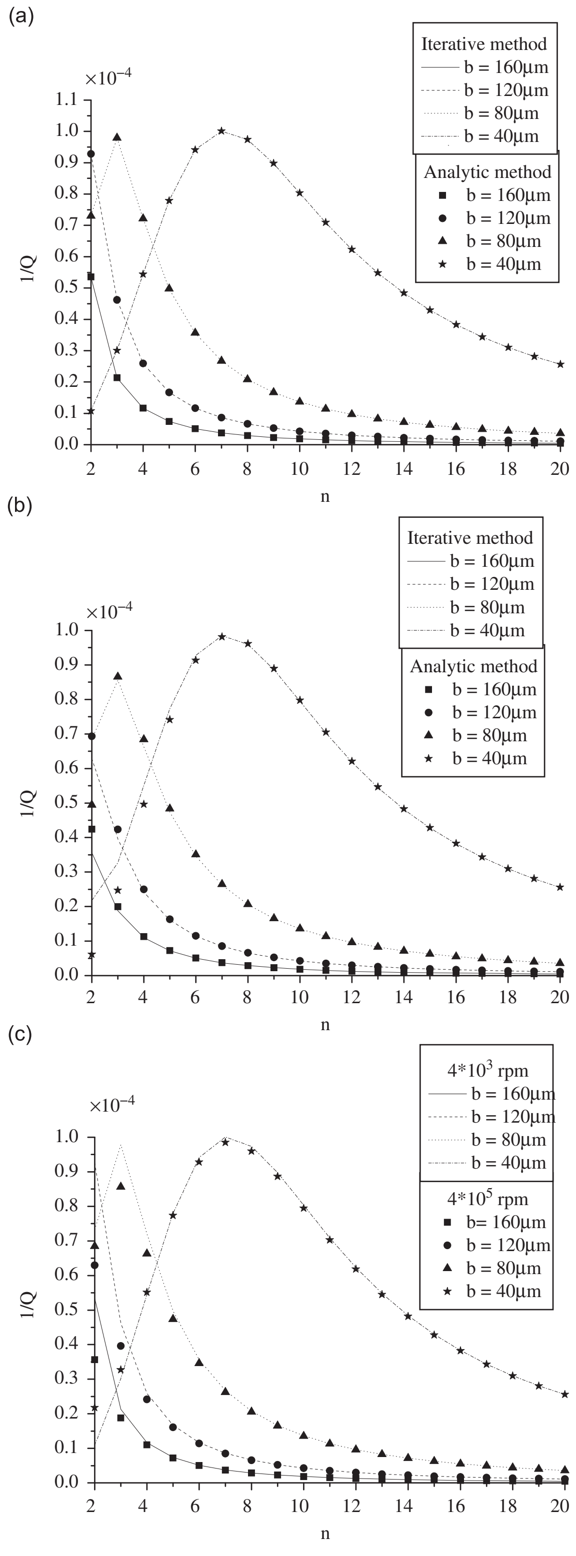

Fig. 3. Inverse of the Q-factor with geometry $a = 3$ mm. (a) $\Omega = 4 \times 10^3$ rpm. (b) $\Omega = 4 \times 10^5$ rpm. (c) Comparison of the iterative results between $\Omega = 4 \times 10^3$ and $\Omega = 4 \times 10^5$ rpm.

Table 3
Q-factor with/without rotation for various temperatures ($a = 3\,\text{mm}$, $b = 120\,\mu\text{m}$).

	Temperature (K)				
	240	258	298	320	348
$\alpha(\times 10^{-6}\,\text{K}^{-1})$	1.99	2.24	2.60	2.85	3.06
$C_v(\times 10^{6}\,\text{Jm}^{-3}\,\text{K}^{-1})$	1.51	1.52	1.64	1.68	1.73
$\chi(\times 10^{-5}\,\text{m}^2\,\text{s}^{-1})$	14.3	11.7	8.60	7.92	6.97
Q-factor($\Omega = 0$)$^{\text{I}}$	19,632	14,444	10,730	8804	7690
Q-factor($\Omega = 4 \times 10^3$)$^{\text{II}}$	19,698	14,503	10,786	8852	7735
Q-factor($\Omega = 4 \times 10^4$)$^{\text{III}}$	20,293	15,042	11,294	9289	8143
Q-factor($\Omega = 4 \times 10^5$)$^{\text{IV}}$	25,416	19,800	15,887	13,262	11,862
(I–II)/I(%)	−0.341	−0.415	−0.522	−0.557	−0.585
(I–III)/I(%)	−3.372	−4.147	−5.256	−5.521	−5.891
(I-IV)/I(%)	−29.469	−37.091	−48.062	−50.653	−54.252

of the Q-factor increase. This result has the same concept in the previous research work for the beam model [7]. As another example, Fig. 3(b) shows the inverse of the Q-factor for rotating speed $\Omega = 4 \times 10^5$ rpm. The difference of the calculated values between iterative and analytic methods is not so large, can be neglected; this can be inferred from Fig. 2. The results obtained by the iterative method for each rotating speed are depicted simultaneously in Fig. 3(c). The Q-factor with $\Omega = 4 \times 10^5$ rpm is larger than the value with $\Omega = 4 \times 10^3$ rpm for the most range of given dimensions and modes.

Table 3 shows a comparison of the Q-factor between non-rotating and rotating cases with respect to temperature. As the rotating speed of the model increases, the Q-factor considerably increases under the given temperatures. Also, the same results can be seen in Fig. 2 for the forward case within $\Omega = 4 \times 10^5$ rpm. Meanwhile, the Q-factors of the thin silicon ring are inversely proportional to the ambient temperatures for each rotating speed as in Fig. 2. Moreover, as the surrounding temperature increases, the difference between cases of both with and without rotating effect increases according to each rotating speed. These results indicate that the effect of rotation on the Q-factor of a thin silicon ring increases for the higher temperature environment.

4. Conclusions

An analytical expression for the Q-factor due to thermoelastic damping is derived for the rotating thin ring in the present work. The result from the analytic method is compared to that of the iterative method to verify the expression of the analytic Q-factors. Furthermore, the influences of geometry, rotational speed and ambient temperature on Q-factors are examined.

It can be shown that the Q-factor of the circular silicon ring is in inverse proportion to the surrounding temperatures. Moreover, as the dimensions of the thin ring become smaller, thermoelastic damping considerably increases. Finally, because the effect of rotation on the Q-factor of a thin ring is significantly important in a high temperature environment and at a very large rotational speed, it should not be neglected.

Acknowledgement

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References

[1] J.P. Gorman, Finite element model of thermoelastic damping in MEMS, M.S Dissertation, Materials Science and Engineering, Massachusetts Institute of Technology, 2002.
[2] S.D. Senturia, *Microsystem Design*, Kluwer Academic Publishers, Boston, 2001.
[3] Z. Hao, A. Erbil, F. Ayazi, An analytical model for support loss in micromachined beam resonators with in-plane vibrations, *Sensors and Actuators A* 109 (2003) 156–164.
[4] A.H. Nayfeh, M.I. Younis, A new approach to the modeling and simulation of flexible microstructures under the effect of squeeze-film damping, *Journal of Micromechanics and Microengineering* 14 (2004) 170–181.
[5] A.H. Nayfeh, M.I. Younis, Modeling and simulations of thermoelastic damping in microplates, *Journal of Micromechanics and Microengineering* 14 (2004) 1711–1717.
[6] C. Zener, Internal friction in solids: I. theory of internal friction in reeds, *Physical Review* 52 (1937) 230–235.
[7] R. Lifshitz, M.L. Roukes, Thermoelastic damping in micro- and nanomechanical systems, *Physical Review B* 61 (2000) 5600–5609.
[8] A. Duwel, M. Weinstein, J. Gorman, J. Borenstein, P. Ward, Quality factors of MEMS gyros and the role of thermoelastic damping, in: Proceedings of the Institute of Electrical and Electronics Engineers, 2002.

[9] Z.F. Khisaeva, M. Ostoja-Starzewski, Thermoelastic damping in nanomechanical resonators with finite wave speeds, *Journal of Thermal Stresses* 29 (2006) 201–216.
[10] S.J. Wong, C.H.J. Fox, S. McWilliam, C.P. Fell, R. Eley, A preliminary investigation of thermoelastic damping in silicon rings, *Journal of Micromechanics and Microengineering* 14 (2004) S108–S113.
[11] Y.B. Yi, Geometric effects on thermoelastic damping in MEMS resonators, *Journal of Sound and Vibration* 309 (2008) 588–599.
[12] Z. Hao, Thermoelastic damping in the contour-mode vibrations of micro- and nano-electromechanical circular thin-plate resonators, *Journal of Sound and Vibration* 313 (2008) 77–96.
[13] P. Lu, H.P. Lee, C. Lu, H.B. Chen, Thermoelastic damping in cylindrical shells with application to tubular oscillator structures International, *Journal of Mechanical Sciences* 50 (2008) 501–512.
[14] L.D. Landau, E.M. Lifshitz, *Theory of Elasticity*, Oxford, Pergamon, 1970.
[15] S.J. Wong, C.H.J. Fox, S. McWilliam, Thermoelastic damping of the inplane vibration of thin silicon rings, *Journal of Sound and Vibration* 293 (2006) 266–285.
[16] W. Soedel, *Vibrations of Shells and plates*, Marcel Dekker, New York, 2004.